Exercise 42

Prove, using Definition 6, that $\lim_{x\to -3} \frac{1}{(x+3)^4} = \infty$.

Solution

According to Definition 6, proving this infinite limit is logically equivalent to proving that

$$\text{if} \quad 0 < |x - (-3)| < \delta \qquad \text{then} \qquad \frac{1}{(x+3)^4} > M,$$

where M is any positive number. Start by working backwards, looking for a number δ that's greater than $|x+3|$.

$$\frac{1}{(x+3)^4} > M$$

$$1 > M(x+3)^4$$

$$\frac{1}{M} > (x+3)^4$$

$$\sqrt[4]{\frac{1}{M}} > \sqrt[4]{(x+3)^4}$$

$$\frac{1}{\sqrt[4]{M}} > |x+3|$$

Choose $\delta = \frac{1}{\sqrt[4]{M}}$. The hypothesis then becomes

$$0 < |x - (-3)| < \delta$$

$$|x+3| < \delta$$

$$|x+3| < \frac{1}{\sqrt[4]{M}}$$

$$(x+3)^4 < \frac{1}{M}$$

$$M(x+3)^4 < 1$$

$$M < \frac{1}{(x+3)^4}.$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to -3} \frac{1}{(x+3)^4} = \infty.$$